A NEW GENERALIZED LAGUERRE-GAUSS COLLOCATION SCHEME FOR NUMERICAL SOLUTION OF GENERALIZED FRACTIONAL PANTOGRAPH EQUATIONS

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Received March 28, 2014

The manuscript is concerned with a generalization of the fractional pantograph equation which contains a linear functional argument. This type of equation has applications in many branches of physics and engineering. A new spectral collocation scheme is investigated to obtain a numerical solution of this equation with variable coefficients on a semi-infinite domain. This method is based upon the generalized Laguerre polynomials and Gauss quadrature integration. This scheme reduces solving the generalized fractional pantograph equation to a system of algebraic equations. Numerical results indicating the high accuracy and effectiveness of this algorithm are presented.

Key words: Functional differential equations; Fractional pantograph equation; Collocation method; Generalized Laguerre-Gauss quadrature; Generalized Laguerre polynomials.

PACS: 03.65.Db, 02.30.Hq.

1. INTRODUCTION

To investigate the behaviors of real world phenomena especially the ones which can not be described fully by the classical methods and techniques we have to invent new mathematical tools or to use the existing non-classical ones. Therefore, the fractional calculus, which means the calculus of derivatives and integral of any order, started to be used as a powerful tool in various disciplines from science and engineering [1]-[3]. The models involving fractional derivatives and integrals have memory, therefore it has proven to be suitable for the description of memory and hereditary properties of various processes [4]-[14]. For some recent developments on this subject, see [15]-[23].

The pantograph equation is a kind of delay differential equations. Pantograph equation was studied by many authors numerically and analytically. Yusufoglu [24] proposed an efficient algorithm for solving generalized pantograph equations with linear functional argument. Yang and Huang [25] presented a spectral-collocation method for fractional pantograph delay-integrodifferential equations. The authors of [26] investigated an exponential approximation to obtain an approximate solution of generalized pantograph-delay differential equations. Furthermore, the authors of [27] proposed a new collocation scheme based on Bernoulli operational matrix for numerical solution of generalized pantograph equation, see also [28]. Recently, in [29], Doha et al. proposed and developed a new Jacobi rational-Gauss collocation method for solving the generalized pantograph equations on a semi-infinite domain.

The main objective of this paper is to propose a new approach to approximate the solution of the generalized fractional Pantograph equation on a semi-infinite interval by using the generalized Laguerre polynomials. The generalized Laguerre spectral collocation (GLC) approximation, which is more reliable, is employed to obtain approximate solution of the form \( u_N(x) \), for the generalized fractional pantograph equation of order \( \nu \) \((m-1 < \nu < m)\) and \( m \) initial conditions. Also, we utilize the \((N-m+1)\) nodes of the generalized Laguerre-Gauss quadrature on the interval \((0, \infty)\), as suitable collocation points. The resulting equations together with those obtained from initial conditions consist of a system of \((N+1)\) algebraic equations which may be solved by any standard numerical scheme. Numerical results are exhibiting the usual exponential convergence behavior of spectral approximations.

This manuscript is organized as follows. We present some definition of fractional calculus needed hereafter in the next section. In Section 3, we introduce the generalized Laguerre polynomials and the interpolation operator of these polynomials on the half line, and in Section 4, the way of constructing the collocation method for solving generalized fractional pantograph equation is implemented using the generalized Laguerre polynomials. The numerical results exhibiting the accuracy and efficiency of our proposed spectral algorithms are introduced in Section 5.

2. BASIC DEFINITIONS

The two most commonly used definitions are the Riemann-Liouville operator and the Caputo operator [30]. We give some definitions and properties of fractional derivatives.

**Definition 2.1.** The Riemann-Liouville fractional integral operator of order \( \nu \) \((\nu > 0)\) is defined as

\[
J^\nu f(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t) dt, \quad \nu > 0, \quad x > 0,
\]

\[
J^0 f(x) = f(x).
\]

(1)
Definition 2.2. The Caputo fractional derivatives of order $\nu$ is defined as

$$D^{\nu} f(x) = J^{m-\nu} D^m f(x) = \frac{1}{\Gamma(m-\nu)} \int_0^x (x-t)^{m-\nu-1} \frac{d^m}{dt^m} f(t) dt,$$

$m - 1 < \nu < m, \ x > 0,$

where $D^m$ is the classical differential operator of order $m$.

For the Caputo derivative we have

$$D^{\nu} C = 0, \quad (C \text{ is a constant}),$$

where $\left\lceil \nu \right\rceil$ and $\left\lfloor \nu \right\rfloor$ are the ceiling and floor functions respectively, while $N = \{1, 2, \ldots\}$ and $N_0 = \{0, 1, 2, \ldots\}$.

The Caputo’s fractional differentiation is a linear operation, similar to the integer-order differentiation

$$D^{\nu}(\lambda f(x) + \mu g(x)) = \lambda D^{\nu} f(x) + \mu D^{\nu} g(x),$$

where $\lambda$ and $\mu$ are constants.

3. GENERALIZED LAGUERRE POLYNOMIALS INTERPOLATION

In this section, we present the properties of generalized Laguerre polynomials that will be useful to construct the collocation scheme.

3.1. GENERALIZED LAGUERRE POLYNOMIALS

We recall below some relevant properties of the generalized Laguerre polynomials [31, 32]. Now, let $\Lambda = (0, \infty)$ and $w^{(\alpha)}(x) = x^\alpha e^{-x}$ be a weight function on $\Lambda$ in the usual sense. Define

$$L^2_{w^{(\alpha)}}(\Lambda) = \{ v \mid v \text{ is measurable on } \Lambda \text{ and } ||v||_{w^{(\alpha)}} < \infty \},$$

equipped with the following inner product and norm

$$(u,v)_{w^{(\alpha)}} = \int_{\Lambda} u(x) v(x) w^{(\alpha)}(x) \, dx, \quad ||v||_{w^{(\alpha)}} = (v,v)_{w^{(\alpha)}}^{\frac{1}{2}}.$$  

The generalized Laguerre polynomials (GLP) $L_n^{(\alpha)}(x)$ are the $n$th eigenfunction of the Sturm-Liouville problem [31]:

$$x \frac{d^2}{dx^2} L_n^{(\alpha)}(x) + (\alpha + 1 - x) \frac{d}{dx} L_n^{(\alpha)}(x) + n L_n^{(\alpha)}(x) = 0, \quad x \in \Lambda, n = 0, 1, 2, \ldots.$$  

(6)
Let $\Gamma(x)$ be the Gamma function, then it is to be noted that

$$D^q L_i^{(\alpha)}(0) = (-1)^q \sum_{j=0}^{i-q} \frac{(i-j-1)!}{(q-1)!(i-j-q)!} L_j^{(\alpha)}(0), \quad i \geq q,$$

(7)

where $L_j^{(\alpha)}(0) = \frac{\Gamma(j+\alpha+1)}{\Gamma(\alpha+1)}$, will be of important use later, for treating the initial conditions of the given FDEs.

The GLP are defined with the following recurrence formula:

$$L_0^{(\alpha)}(x) = 1, \quad L_1^{(\alpha)}(x) = 1 + \alpha - x,$$

$$L_{i+1}^{(\alpha)}(x) = \frac{1}{i+1} \left[ (2i+\alpha+1-x)L_i^{(\alpha)}(x) - (i+\alpha)L_{i-1}^{(\alpha)}(x) \right], \quad i = 1, 2, \ldots,$$

where $\alpha > -1$ and $i > 1$. The set of generalized Laguerre polynomials is the $L_{w^{(\alpha)}}^2(\Lambda)$-orthogonal system, namely

$$\int_0^\infty L_j^{(\alpha)}(x)L_k^{(\alpha)}(x)w^{(\alpha)}(x)\,dx = h_k \delta_{jk},$$

(8)

where $\delta_{jk}$ is the Kronecker function and $h_k = \frac{\Gamma(k+\alpha+1)}{k!}$. The analytical form of generalized Laguerre polynomials of degree $i$ on $\Lambda$, is given by (see e.g. [33])

$$L_i^{(\alpha)}(x) = \sum_{k=0}^{i} (-1)^k \frac{\Gamma(i+\alpha+1)}{\Gamma(k+\alpha+1)} \frac{1}{(i-k)!} \frac{1}{k!} \frac{1}{\Gamma(\alpha+1)} x^k, \quad i = 0, 1, \ldots.$$

(9)

Since the analytic form of the generalized Laguerre polynomials $L_i^{(\alpha)}(x)$ of degree $i$ is given by (9), the use of Eqs. (4), (5) and (9) yields

$$D^\nu L_i^{(\alpha)}(x) = \sum_{k=0}^{i} (-1)^k \frac{\Gamma(i+\alpha+1)}{(i-k)!} \frac{1}{k!} \frac{1}{\Gamma(\alpha+1)} D^\nu x^k$$

$$= \sum_{k=[\nu]}^{i} (-1)^k \frac{\Gamma(i+\alpha+1)}{(i-k)!} \frac{1}{\Gamma(k-\nu+1)} \frac{1}{\Gamma(k+\alpha+1)} x^{k-\nu},$$

(10)

$$i = [\nu], \ldots, N.$$

Now, approximate $x^{k-\nu}$ by $N+1$ terms of generalized Laguerre series, we have

$$x^{k-\nu} = \sum_{j=0}^{N} b_j L_j^{(\alpha)}(x),$$

(11)
where $b_j$ is given from (14) with $u(x) = x^{k-\nu}$, and

$$b_j = \sum_{\ell=0}^{j} (-1)^\ell \frac{j! \Gamma(k-\nu+\alpha+\ell+1)}{(j-\ell)! (\ell)! \Gamma(\ell+\alpha+1)}, \quad (12)$$

Accordingly

$$D^\nu L^{(\alpha)}_i(x) = \sum_{j=0}^{N} S_\nu(i,j) L^{(\alpha)}_j(x), \quad i = [\nu], \cdots, N, \quad (13)$$

where

$$S_\nu(i,j) = \sum_{k=[\nu]}^{i} \sum_{\ell=0}^{j} (-1)^{k+\ell} \frac{j! \Gamma(i+\alpha+1) \Gamma(k-\nu+\alpha+\ell+1)}{(i-k)! (\ell)! \Gamma(k-\nu+1) \Gamma(k+\alpha+1) \Gamma(\alpha+\ell+1)}.$$

Let $u(x) \in L^2_{w^{(\alpha)}}(\Lambda)$, then $u(x)$ may be expressed in terms of generalized Laguerre polynomials as

$$u(x) = \sum_{j=0}^{\infty} a_j L^{(\alpha)}_j(x), \quad (14)$$

$$a_j = \frac{1}{h_k} \int_0^\infty u(x) L^{(\alpha)}_j(x) w^{(\alpha)}(x) dx, \quad j = 0, 1, 2, \ldots.$$

### 3.2. GENERALIZED LAGUERRE POLYNOMIALS INTERPOLATION APPROXIMATION

We will introduce the Laguerre-Gauss-type quadratures, including Laguerre-Gauss and Laguerre-Gauss-Radau. Let $\{x_j^{(\alpha)}, \varpi_j^{(\alpha)}\}$ be the set of generalized Laguerre-Gauss or generalized Laguerre-Gauss-Radau quadrature nodes and weights (see e.g. [31]).

$$\int_\Lambda \phi(x) w^{(\alpha)}(x) dx = \sum_{j=0}^{N} \phi(x_j^{(\alpha)}) \varpi_j^{(\alpha)} \quad (15)$$

- For the generalized Laguerre-Gauss quadrature (GQ): $\{x_j^{(\alpha)}\}$ are the zeros of $L^{(\alpha)}_{i+1}(x)$;

  $$\varpi_j^{(\alpha)} = -\frac{\Gamma(i+\alpha+1)}{(i+1)! L_i^{(\alpha)}(x_j^{(\alpha)}) \partial_x L_i^{(\alpha)}(x_j^{(\alpha)})}$$

  $$= \frac{\Gamma(i+\alpha+1) x_j^{(\alpha)}}{(i+\alpha+1)(i+1)! [L_i^{(\alpha)}(x_j^{(\alpha)})]^2}, \quad 0 \leq j \leq i.$$

- For the generalized Laguerre-Gauss-Radau quadrature (GRQ):
\[ x_0^{(\alpha)} = 0, \{x_j^{(\alpha)}\}_{j=1}^i \text{ are the zeros of } \partial_x L_{i+1}^{(\alpha)}(x); \]

\[ \varpi_0^{(\alpha)} = \frac{(\alpha + 1)^2(\alpha + 1)\Gamma(i + 1)}{\Gamma(i + \alpha + 2)}, \]

\[ \varpi_j^{(\alpha)} = \frac{\Gamma(i + \alpha + 1)}{i!(i + \alpha + 1)[\partial_x L_i^{(\alpha)}(x_j^{(\alpha)})]^2} = \frac{\Gamma(i + \alpha + 1)}{i!(i + \alpha + 1)(L_i^{(\alpha)}(x_j^{(\alpha)}))^2}, \quad 1 \leq j \leq i. \]

Now, let \( N \) be any positive integer. We denote by \( P_N \) the set of all algebraic polynomials of degree at most \( N \). The orthogonal projection \( P_{N,\alpha} : L^2_{w(\alpha)}(\Lambda) \rightarrow P_N \) is defined by

\[ (P_{N,\alpha}v - v, \phi)_{w(\alpha)} = 0, \forall \phi \in P_N. \]

For any integer \( r > 0 \), and \( c \) a generic positive constant which does not depend on \( N \) and any function, we have for any \( v \in L^2_{w(\alpha)}(\Lambda) \) the following approximation result

\[ ||P_{N,\alpha}v - v||_{w(\alpha)} \leq cN^{-\frac{r}{2}}||\partial_x^r v||_{w(\alpha)+r}. \]

4. Generalized Laguerre-Gauss Collocation Method

The main idea of the spectral collocation scheme is to expand the approximate solution of the problem as a finite sum of orthogonal polynomials and then to seek the coefficients in order to minimize the error as much as possible. For spectral collocation method see Refs. [34]-[37]. In this section, we use the generalized Laguerre-Gauss collocation method to solve numerically the following model problem:

\[ D^\nu u(x) = \sum_{j=0}^J \sum_{n=0}^{m-1} p_{jn}(x) D^\gamma_n u(\lambda_{jn} x + \mu_{jn}) + g(x), \quad x \in (0, \infty), \quad (16) \]

subject to

\[ \sum_{n=0}^{m-1} a_{in} u^{(n)}(0) = \lambda_i, \quad i = 0, 1 \ldots m - 1. \quad (17) \]

where \( a_{in}, \lambda_i, \) and \( \lambda_{jn} \) are real or complex coefficients, \( m - 1 < \nu \leq m, \quad 0 < \gamma_0 < \gamma_1 \ldots < \gamma_{m-1} < \nu, \) meanwhile \( p_{jn}(x) \) and \( g(x) \) are given continuous functions in the interval \((0, \infty)\). By using the generalized Laguerre-Gauss collocation method, we can approximate the generalized fractional pantograph equation with variable coefficients on a semi-infinite domain directly, without any artificial boundary and variable transformation.
Let us first introduce some basic notation that will be used in the sequel. We set
\[ S_N(0, \infty) = \text{span}\{L_0^{(\alpha)}(x), L_1^{(\alpha)}(x), \ldots, L_N^{(\alpha)}(x)\}, \]  
(18)
and we define the discrete inner product and norm as follows:
\[ (u,v)_{w_N^{(\alpha)}} = \sum_{j=0}^{N} u(x_j^{(\alpha)}) v(x_j^{(\alpha)}) \omega_j^{(\alpha)}, \quad \|u\|_{w_N^{(\alpha)}} = \sqrt{(u,u)_{w_N^{(\alpha)}}}. \]  
(19)
where \(x_j^{(\alpha)}\) and \(\omega_j^{(\alpha)}\) are the nodes and the corresponding weights of the generalized Laguerre-Gauss quadrature formula on the interval \((0, \infty)\), respectively. Obviously, \((u,v)_{w_N^{(\alpha)}} = (u,v)_{w_N^{(\alpha)}}, \forall u, v \in S_{2N+1}^{2N+1} \).
(20)
Thus, for any \(u \in S_N(0, \infty)\), the norms \(\|u\|_{w_N^{(\alpha)}}\) and \(\|u\|_{w_N^{(\alpha)}}\) coincide.

Associating with this quadrature rule, we denote by \(I_N^{L_T^{(\alpha)}}\) the generalized Laguerre-Gauss interpolation,
\[ I_N^{L_T^{(\alpha)}} u(x_k^{(\alpha)}) = u(x_k^{(\alpha)}), \quad 0 \leq k \leq N. \]
The generalized Laguerre-Gauss collocation method for solving (16) and (17) is to seek \(u_N(x) \in S_N(0, \infty)\), such that
\[ D^{\nu} u(x_k^{(\alpha)}) = \sum_{j=0}^{J} \sum_{n=0}^{m-1} p_{jn}(x_k^{(\alpha)}) D^{\gamma_n} u(\lambda_j \gamma x_k^{(\alpha)} + \mu_j n + g(x_k^{(\alpha)})), \]
\[ k = 0, 1, \ldots, N - m, \]
(21)
\[ \sum_{n=0}^{m-1} a_{i,n} u^{(n)}(0) = \lambda_i, \quad i = 0, 1, \ldots, m - 1. \]

We now derive the algorithm for solving (16) and (17). To do this, let
\[ u_N(x) = \sum_{h=0}^{N} a_h \bar{L}_h^{(\alpha)}(x), \quad a = (a_0, a_1, \ldots, a_N)^T. \]  
(22)
We first approximate \(D^{\nu} u(x)\) and \(D^{\gamma_n} u(x)\), as Eq. (22). By substituting these approximation in Eq. (16), we get
\[ \sum_{h=0}^{N} a_h D^{\nu} \bar{L}_h^{(\alpha)}(x) = \sum_{j=0}^{J} \sum_{n=0}^{m-1} \sum_{h=0}^{N} a_h p_{jn}(x) D^{\gamma_n} \bar{L}_h^{(\alpha)}(\lambda_j \gamma x + \mu_j n + g(x)). \]  
(23)
Making use of (13), we deduce that
\[
\sum_{h=0}^{N} \sum_{f=0}^{M} a_h S_{\nu}(h, f) L_{j}^{(\alpha)}(x) = \sum_{j=0}^{J} \sum_{m=0}^{N} \sum_{h=0}^{M} a_h p_{jn}(x) S_{\gamma_n}(h, f) L_{f}^{(\alpha)}(\lambda_{jn} x + \mu_{jn}) + g(x).
\] (24)

Also, by substituting Eq. (22) in Eq. (17) we obtain
\[
\sum_{n=0}^{m-1} \sum_{f=0}^{M} a_{in} D^{(\alpha)} L_{f}^{(\alpha)}(0) = \lambda_i.
\] (25)

Now, we collocate Eq. (24) at the \((N - m + 1)\) generalized Laguerre-Gauss interpolation points, yields
\[
\sum_{h=0}^{N} \sum_{f=0}^{M} a_h S_{\nu}(h, f) L_{j}^{(\alpha)}(x_k) = \sum_{j=0}^{J} \sum_{m=0}^{N} \sum_{h=0}^{M} a_h p_{jn}(x_k) S_{\gamma_n}(h, f) L_{f}^{(\alpha)}(\lambda_{jn} x_k^{(\alpha)} + \mu_{jn}) + g(x_k).
\] (26)

Next Eq. (25), after using (7), can be written as
\[
\sum_{n=0}^{m-1} \sum_{f=0}^{M} \sum_{l=0}^{f-n} (-1)^{l} a_{in} (f - l - 1)! \Gamma(l + \alpha + 1) (n - 1)! (f - l - n)! \Gamma(\alpha + 1)! = \lambda_i.
\] (27)

Finally, Eq. (26) with relation (27) generate \((N + 1)\) set of algebraic equations which can be solved for the unknown coefficients \(a_j, j = 0, 1, 2, \ldots, N\), by using any standard solver technique.

5. NUMERICAL RESULTS

In this section, two generalized fractional pantograph equations with variable coefficients on a semi infinite interval are solved numerically by the present method to demonstrate its accuracy and capability. All of these problems were performed on the computer using a program written in Mathematica 8.0. The absolute errors in the given tables are the values of \(|u(x) - u_N(x)|\) at selected points.

Example 1. Consider the following generalized fractional pantograph equation with variable coefficients
\[
u^{(1/2)}(x) = -u(x) + x^2 u(x/2) + xu(x/4) + g(x), u(0) = 0, \quad x \in [0, 1],
\] (28)

where
\[
g(x) = \frac{1}{\Gamma\left(-\frac{1}{2}\right)} \int_0^x (x - t)^{-\frac{3}{2}} t^3 e^{-t} dt + x^2 e^{-x} - \frac{x^5}{8} e^{-\frac{x}{2}} - \frac{x^4}{64} e^{-\frac{x}{4}},
\]
Table 1
Absolute errors using GLC method at $N = 22$ for Example 1.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\alpha = 1$</th>
<th>$\alpha = 2$</th>
<th>$\alpha = 3$</th>
</tr>
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<tr>
<td>0.1</td>
<td>$1.427.10^{-4}$</td>
<td>$4.885.10^{-4}$</td>
<td>$1.273.10^{-2}$</td>
</tr>
<tr>
<td>0.2</td>
<td>$1.360.10^{-3}$</td>
<td>$6.152.10^{-3}$</td>
<td>$1.794.10^{-2}$</td>
</tr>
<tr>
<td>0.3</td>
<td>$7.568.10^{-4}$</td>
<td>$5.629.10^{-4}$</td>
<td>$1.888.10^{-2}$</td>
</tr>
<tr>
<td>0.4</td>
<td>$1.405.10^{-4}$</td>
<td>$4.462.10^{-4}$</td>
<td>$1.777.10^{-2}$</td>
</tr>
<tr>
<td>0.5</td>
<td>$2.491.10^{-4}$</td>
<td>$3.310.10^{-4}$</td>
<td>$1.589.10^{-2}$</td>
</tr>
<tr>
<td>0.6</td>
<td>$3.501.10^{-4}$</td>
<td>$2.503.10^{-4}$</td>
<td>$1.422.10^{-2}$</td>
</tr>
<tr>
<td>0.7</td>
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<td>$2.156.10^{-4}$</td>
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</tr>
<tr>
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</tr>
<tr>
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<tr>
<td>1.0</td>
<td>$8.880.10^{-4}$</td>
<td>$3.430.10^{-3}$</td>
<td>$1.473.10^{-2}$</td>
</tr>
</tbody>
</table>

Fig. 1 – Graph of exact solution and approximate solution for $\alpha = 0$ at $N = 22$ for Example 1.
and the exact solution is given by \( u(x) = x^3 e^{-x} \).

In Table 1, we list the absolute errors obtained by the generalized Laguerre collocation method, with different values of \( \alpha \) and \( N = 22 \). The numerical results presented in this table show that the results are accurate. In order to compare the present method with the analytic solution, the resulting graph of Eq. (28) is shown in Fig. 1.

**Example 2.** Consider the following generalized fractional pantograph equation

\[
 u^{(5/2)}(x) = -u(x) - u(x - 0.5) + g(x), \\
 u(0) = 0, \ u'(0) = 0, \ u''(0) = 0, \ x \in [0, 1],
\]

(29)
where
\[ g(x) = \frac{\Gamma(4)}{\Gamma\left(\frac{7}{2}\right)} x^{\frac{1}{2}} + x^3 + (x - 0.5)^3, \]
and the exact solution is given by \( u(x) = x^3 \).

In Table 2, we list the absolute errors obtained by the generalized Laguerre collocation method, with several values of \( \alpha \) and at \( N = 22 \). Meanwhile, Fig. 2 plots the spectral solution with \( \alpha = 1 \) at \( N = 22 \) and exact solution, which are found to be in excellent agreement. This numerical experiment demonstrates the utility of the method.

6. CONCLUSION

In this paper, we have proposed a new numerical algorithm to solve the generalized fractional pantograph equation with variable coefficients on a semi-infinite domain which contains a linear functional argument. The generalized Laguerre collocation approximation based on generalized Laguerre-Gauss interpolation nodes was developed to solve this problem. A number of collocation techniques can be obtained as special cases from the proposed technique. Numerical results were given to demonstrate the accuracy and applicability of the proposed method.

Acknowledgements. This paper was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, under grant no. (4-135-35-RG). The authors, therefore, acknowledge with thanks DSR technical and financial support.

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