LOCALIZED NONLINEAR MATTER WAVES IN ONE-DIMENSIONAL
BOSE-EINSTEIN CONDENSATES WITH SPATIOTEMPORALLY
MODULATED TWO- AND THREE-BODY INTERACTIONS

DENG-SHAN WANG¹, YUSHAN XUE², ZHIFEI ZHANG³

¹School of Applied Science, Beijing Information
Science and Technology University, Beijing 100192, China
 Corresponding author E-mail: wangdsh1980@163.com
²School of Statistics and Mathematics, Central University of Finance and Economics,
Beijing 100081, China
³Department of Mathematics, Huazhong University of Science and
Technology, Wuhan, 430074, China
 Corresponding author E-mail: zfzhangwh@163.com

Received February 4, 2016

In this paper, we study the localized nonlinear matter waves in one-dimensional Bose-Einstein condensates with spatiotemporally modulated two- and three-body interactions and time-dependent external potentials. By means of similarity transformation, two exact solutions of the one-dimensional cubic-quintic Gross-Pitaevskii equation are obtained explicitly. It is demonstrated that both attractive two- and three-body interactions and attractive two-body plus repulsive three-body interactions support localized nonlinear matter waves in Bose-Einstein condensates trapped in time-dependent external potentials. The dynamics of the localized nonlinear matter wave is analyzed and some breathing and quasi-breathing solutions are found.

Key words: Two- and three-body interactions, external potential, localized nonlinear matter wave, breathing solution, quasi-breathing solution.

PACS: 03.75.Hh, 05.45.Yv, 67.85.Bc.

1. INTRODUCTION

The experimental realization of Bose-Einstein condensation (BEC) in ultracold atomic gases [1]-[3] has opened many hot topics in theoretical research on the properties of Bose gases such as vortex lattice [4], magnetic monopoles [5], skyrmions [6], bright and dark solitons [7, 8] and localized nonlinear matter waves [9, 10]. The properties of BEC [11, 12] including their shape, stability, and collective nonlinear excitations are determined by the external potentials and the s-wave scattering lengths. Generally speaking, the harmonic potential is popular in the real experiments and a remarkable way to adjust scattering length is to tune an external optical or magnetic field in the vicinity of a Feshbach resonance [13].

In the past two decades, techniques for adjusting the external potentials and scattering lengths globally have been crucial to many theoretical and experimental achievements [7, 9, 14, 15]. One can choose the confinement frequencies of

the external potentials to be time dependent so as to control the shape of the Bose gases. Moreover, the scattering lengths can be manipulated by time-modulated [7], space-modulated [9, 16] or spatiotemporally modulated nonlinearities [10, 17]. In the framework of mean-field approximation, BECs are described by the well-known Gross-Pitaevskii (GP) equation [9, 10, 16–18], which is a cubic nonlinear Schrödinger equation with external potential in which the cubic nonlinearity models the two-body interatomic interactions appropriate for dilute gases. However, the GP equation with cubic nonlinearity can only be used to study the Bose gases at low densities. For higher densities, three-body interactions play a key role and a quintic nonlinearity should be introduced. Thus a more accurate treatment of the mean-field approximation of a dense condensate will need to account for both two- and three-body elastic collisions [19–22]. Correspondingly, the GP equation with cubic-quintic nonlinearity is necessary to achieve the description of BECs with two- and three-body interactions.

So far, some studies on BECs with two- and three-body interactions are done [21]-[24]. For example, Paz-Alonso and Michinel [23] demonstrated the generation of stable vortex lattices in light condensates by numerical simulations. Wang and Li [21] investigated the profiles and stability of the localized nonlinear matter waves in a BEC with spatially inhomogeneous two- and three-body interactions. Avelar et al. [24] derived the soliton solutions of GP equation with cubic-quintic nonlinearity based on similarity transformation. In the present paper, we continue to explore the nonlinear matter waves in a BEC with spatiotemporally modulated two- and three-body interactions. Especially, we will investigate the dynamics of BEC with distributed gain/loss effect and time-dependent external potential.

2. THE THEORETICAL MODEL AND ITS EXACT SOLUTIONS

In the mean-field approximation, the properties of an atomic BEC with two- and three-body interactions trapped in a harmonic potential is governed by the three-dimensional (3D) GP equation with cubic-quintic nonlinearities

\[ i\hbar \frac{\partial \varphi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \varphi + V_{\text{ext}} \varphi + G_2 |\varphi|^2 \varphi + G_3 |\varphi|^4 \varphi, \]

where \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \), \( r^2 = y^2 + z^2 \) denotes the radial distance and \( \theta \) is the azimuthal angle, \( m \) is the atom mass, the wave function \( \varphi = \varphi(r, t) \) is normalized by the total particle number \( N = \int d\mathbf{r} |\varphi|^2 \), \( V_{\text{ext}} = (m/2)(\omega_\perp^2 r^2 + \omega_x^2 x^2) \) is an external trapping potential with \( \omega_\perp \) and \( \omega_x \) the trapping frequencies in the transverse and longitudinal directions (\( \omega_\perp \gg \omega_x \)).
Localized nonlinear matter waves in one-dimensional Bose-Einstein condensates

respectively, and parameters $G_2$ and $G_3$ are the nonlinear coefficients corresponding to the two- and three-body interactions, respectively. Here the two-body interaction $G_2 = 4\pi \hbar^2 a_s/m$ represents the strength of interatomic interaction characterized by the s-wave scattering length $a_s$, which is positive (negative) for repulsive (attractive) condensates consisting of, e.g., $^{23}\text{Na}$ or $^{87}\text{Rb}$ ($^{85}\text{Rb}$ or $^7\text{Li}$) atoms. The three-body interaction is $G_3 = (12\pi \hbar^2 a_s^3/m)[d_1 + d_2 \tan(s_0 \ln(|a_s|/|a_0|) + \pi/2)]$ [25], where $s_0$ is a dimensionless constant and $d_1, d_2 < 0, a_0$ have been determined numerically by Braaten et al. [26]. Especially, for $^{87}\text{Rb}$ atoms the interaction parameters $G_2 \approx 5\hbar \times 10^{-11} \text{ cm}^3/\text{s}$ and $G_3 \approx 4\hbar \times 10^{-26} \text{ cm}^3/\text{s}$. In this paper, as in [10, 17], we consider a collisionally spatiotemporally inhomogeneous condensate, i.e., a spatiotemporally varying scattering length according to $a_s = a_s(x,t)$. Moreover, in the case of the strong anisotropy of the trap one can average the three-dimensional interaction over the radial density profile to reduce the BEC system (1) to an effective one-dimensional one. To do so, consider the solution of the three-dimensional GP equation (1) in the form of

$$\varphi(x,y,z,t) = \varphi(x,r,t) = \varphi_0(r) \psi(x,t),$$

(2)

where $\varphi_0(r) = \sqrt{1/\pi a_\perp^2} e^{-r^2/2a_\perp^2}$ with $a_\perp = \sqrt{\hbar/m\omega_\perp}$ is the ground state of the radial linear equation

$$-\frac{\hbar^2}{2m} \nabla^2 \varphi_0 + \frac{1}{2} m\omega_\perp^2 r^2 \varphi_0 = \hbar \omega_\perp \varphi_0.$$

(3)

Substituting Eq. (2) with Eq. (3) in Eq. (1) and multiplying the two sides by $\varphi_0(r)$ and integrating with respect to the transverse variable $r$, we finally get the one-dimensional (1D) cubic-quintic GP equation as

$$i\hbar \psi_t = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi + \frac{G_2}{2\pi a_\perp^2} |\psi|^2 \psi + \frac{G_3}{3\pi^2 a_\perp^4} |\psi|^4 \psi,$$

(4)

where the external potential is $V(x) = m\omega_\perp^2 x^2/2$, which can be reduced to a dimensionless form as

$$i\psi_t = -\frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi + g_2 |\psi|^2 \psi + g_3 |\psi|^4 \psi,$$

(5)

where $V(x) = \omega_x^2 x^2/2$ with $\omega = \omega_x/\omega_\perp$ determines the magnetic trap strength, the interaction parameters are $g_2 = mG_2/2\pi\hbar^2$, $g_3 = m^2\omega_\perp G_3/3\pi^2\hbar^3$ and the length and time are measured in units of $a_\perp = \sqrt{\hbar/m\omega_\perp}$ and $\omega_\perp^{-1}$, respectively.

It is remarked that some other assumptions of wave function $\varphi(x,y,z,t)$ in Eq. (2) can give rise to different 1D GP equations. In previous works [27–29] have been derived some effective non-polynomial wave equations starting from the 3D GP equation with cubic nonlinearity. Thus following the procedure in [27–29] one
can derive the corresponding non-polynomial wave equations based on the 3D cubic-quintic GP equation (1).

In what follows, however, we focus on the following 1D cubic-quintic GP equation with gain/loss rather than the 1D cubic-quintic GP equation (5)

\[
i_{t} = \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + V(x,t) \psi + g_2 |\psi|^2 \psi + g_3 |\psi|^4 \psi + i \gamma \psi,
\]

where \( g_2 \) and \( g_3 \) are spatiotemporally modulated cubic and quintic nonlinearities, \( i.e. \) they are functions of \((x,t)\), \( V(x,t) \) is a space- and time-dependent external potential and \( \gamma \) is function of time \( t \). The parameter \( \gamma \) denotes the distributed gain/loss effect on a condensate \( \text{(30)} \), where the small gain (loss) coefficient \( \gamma > 0 \) (\( \gamma < 0 \)) means that in a steady state the condensate absorbs (releases) some atoms from (into) the background thermal cloud by optically pumping atoms from the external cold atomic-beam source. In general, the gain/loss parameter \( \gamma \) can be constant as the cubic GP gain equation studied by Yuce and Kilic \( \text{(31)} \). Especially, the gain parameter \( \gamma \) accounts for the feeding that can be controlled by using a reservoir filled with a large amount of the condensate to which the trap is weakly coupled \( \text{(32, 33)} \).

Based on similarity transformation, we investigate the localized nonlinear matter wave solutions of the equation (6). In doing so, we assume

\[
\psi(x,t) = \rho \Psi(X)e^{i\phi},
\]

where \( \rho = \rho(x,t), X = X(x,t) \) and \( \phi = \phi(x,t) \). Substituting Eq. (7) into (6), letting \( \Psi = \Psi(X) \) and solving the ordinary differential equation (ODE)

\[
\frac{\partial^2 \Psi}{\partial X^2} + \sigma_1 \Psi + \sigma_2 \Psi^3 + \sigma_3 \Psi^5 = 0,
\]

it is easy to find that functions \( \rho, X, \phi \) solve the following set of partial differential equations:

\[
2 \rho_x X_x + X_{xx} \rho = 0, \quad X_t + \phi_x X_x = 0, \quad 2 \rho^4 g_3 + \sigma_3 X_x^2 = 0, \quad 2 \rho^2 g_2 + \sigma_2 X_x^2 = 0,
\]

and

\[
\sigma_1 \rho X_x^2 + 2 \phi_t \rho - \rho_{xx} + \phi_x^2 \rho + 2 V(x,t) \rho = 0, \quad 2 \rho_t - 2 \gamma \rho + \phi_{xx} \rho + 2 \rho_x \phi_x = 0.
\]

Solving this set of partial differential equations, we have

\[
X = \text{erf} (\delta x), \quad \phi = \tau_1 x^2 + \tau_2, \quad \rho = \eta e^{\delta^2 x^2 / 2}, \quad V = \frac{\omega^2}{2} x^2 - \frac{2 \sigma_1 \delta^2 e^{-2 \delta^2 x^2}}{\pi}, \quad \text{and}
\]

\[
g_2 = -\frac{2 \sigma_2 \delta^2 e^{-3 \delta^2 x^2}}{\pi \eta^2}, \quad g_3 = -\frac{2 \sigma_3 \delta^2 e^{-4 \delta^2 x^2}}{\pi \eta^4},
\]
where \( \text{erf}(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-\tau^2} d\tau \) is called the error function. \( \delta, \eta, \tau_1, \tau_2, \omega, \) and \( \gamma \) are functions of time \( t \), which satisfy

\[
\delta_{tt} \delta - 2 \delta_t^2 + \delta^6 - \omega^2 \delta^2 = 0, \quad (12)
\]

\[
\tau_1 = -\frac{\delta_t}{2\delta}, \quad \tau_2 = \int \frac{\delta^2}{2} dt, \quad 2 \gamma \eta \delta - 2 \eta_t \delta + \eta \delta_t = 0. \quad (13)
\]

For the second-order ODE (12), setting \( \delta = 1/y(t) \) we have

\[
y_{tt} + \omega^2 y = 1/y^3, \quad (14)
\]

which was studied early by Ermakov in 1880 \[34\]. The general solution to this equation is \[35\]

\[
y(t) = \left[ a y_1^2 + b y_2^2 + 2 c y_1 y_2 \right]^{1/2}, \quad (15)
\]

where \( a, b, c \) are constants satisfying \( ab - c^2 = 1/W^2 \), and the Wronskian \( W = y_1 y_{2t} - y_2 y_{1t} \) with \( (y_1, y_2) \) being two linearly independent solutions of homogeneous ODE

\[
y_{tt} + \omega^2 y = 0. \quad (16)
\]

Thus the general solution of the two-order ODE (12) is

\[
\delta(t) = \left[ a y_1^2 + b y_2^2 + 2 c y_1 y_2 \right]^{-1/2}. \quad (17)
\]

For the ODE \[2 \gamma \eta \delta - 2 \eta_t \delta + \eta \delta_t = 0 \] in Eq. (13), it is easy to see that

\[
\gamma = [\ln(\eta/\sqrt{\delta})]_t, \quad (18)
\]

which denotes that

\[
\eta = \sqrt{\delta} e^{\int \gamma dt}. \quad (19)
\]

Moreover, the nonlinear ODE (8) has two groups of exact solutions for two kinds of constraints on the parameters \( \sigma_1, \sigma_2 \) and \( \sigma_3 \):

When

\[
\sigma_3 = \sigma_2^2 \left( \sigma_1 - \nu_1^2 + 2 \nu_1^2 k^2 \right) \left( \nu_1^2 k^2 - \sigma_1 - 2 \nu_1^2 \right) \left( \nu_1^2 - \sigma_1 + \nu_1^2 k^2 \right) / 4 \nu_1^2,
\]

with \( \nu_1 = \nu_1^4 - \nu_1^4 k^2 - \sigma_1^2 + \nu_1^4 k^4 \), the nonlinear ODE (8) has an exact Jacobi elliptic function solution as

\[
\Psi(X) = \frac{\sqrt{2} \text{cn} \left( \nu_1 X, k \right)}{\sqrt{\sigma_2 \left( \nu_1^2 k^2 - \sigma_1 - 2 \nu_1^2 k^2 \right) \text{sn}^2 \left( \nu_1 X, k \right) + \nu_1^2 (2 - k^2) + \sigma_1 / \nu_1}}. \quad (20)
\]

When

\[
\sigma_3 = \sigma_2^2 \left( \sigma_1 - \nu_2^2 + 2 \nu_2^2 k^2 \right) \left( \nu_2^2 k^2 - \sigma_1 - 2 \nu_2^2 \right) \left( \nu_2^2 - \sigma_1 + \nu_2^2 k^2 \right) / 4 \nu_2^2,
\]
Fig. 1 – (Color online) The shapes of the external potential $V(x,t)$ in Eq. (24) with (25) for parameter $\epsilon = 0$. (a)-(c) show the spatiotemporal structures of $V(x,t)$ for parameter $\sigma_1 = 0$, $-2$, and $2$, respectively. (d) demonstrates the spatial structures of $V(x,t)$ in (a)-(c) at $t = 0$ for $\sigma_1 = 0$ (purple line), $\sigma_1 = -2$ (dashed blue line) and $\sigma_1 = 2$ (dashed red line).

with $\nu_2 = \nu_2^4 - \nu_2^4 k^2 - \sigma_1^2 + \nu_2^4 k^4$, the nonlinear ODE (8) has the other exact Jacobi elliptic function solution

$$\Psi(X) = \frac{\sqrt{2}\text{sn}(\nu_2 X,k)}{\sqrt{\sigma_2[(\sigma_1 - \nu_2^2 - \nu_2^2 k^2)\text{sn}^2(\nu_2 X,k) + 3\nu_2^2]/\nu_2}}, \quad (21)$$

where $k$ is the modulus of Jacobi elliptic functions with $0 \leq k \leq 1$, and $\nu_1$ and $\nu_2$ are nonzero constants.

Therefore, we conclude this section that when the external potential is $V(x,t) = \omega^2 x^2/2 - 2\sigma_1 \delta^2 \exp(-2\delta^2 x^2)/\pi$, the two-body and three-body interaction parameters $g_1, g_2$ satisfy Eq. (11) and the gain/loss coefficient $\gamma$ satisfies the constraint in Eq. (19), the 1D cubic-quintic GP equation with gain/loss in Eq. (6) owns two exact localized nonlinear matter wave solutions

$$\psi(x,t) = \frac{\sqrt{2}\text{sech}^{\gamma} t e^{i\delta^2 x^2/2}\text{cn}(\nu_1 \text{erf}(\delta x), k)e^{i(\nu_1^2 x^2 + \nu_2^2 x^2)}}{\sqrt{\sigma_2[(\sigma_1 - \nu_1^2 - 2\nu_1^2 k^2)\text{sn}^2(\nu_1 \text{erf}(\delta x), k) + \nu_1^2(2-k^2) + \sigma_1]/\nu_1}}, \quad (22)$$

$$\psi(x,t) = \frac{\sqrt{2}\text{sech}^{\gamma} t e^{i\delta^2 x^2/2}\text{sn}(\nu_2 \text{erf}(\delta x), k)e^{i(\nu_1^2 x^2 + \nu_2^2 x^2)}}{\sqrt{\sigma_2[(\sigma_1 - \nu_1^2 - 2\nu_1^2 k^2)\text{sn}^2(\nu_2 \text{erf}(\delta x), k) + 3\nu_2^2]/\nu_2}}, \quad (23)$$

where the functions $\delta, \tau_1, \tau_2, \eta, \omega$, and $\gamma$ satisfy Eqs. (12)-(19).
3. DYNAMICS OF THE LOCALIZED NONLINEAR MATTER WAVES

In this Section, we investigate the dynamics of the exact localized nonlinear matter wave solutions (22)-(23) of the 1D cubic-quintic GP equation (6). We also point out how to control the dynamics of the localized nonlinear waves in the one-dimensional BEC with spatiotemporally modulated two- and three-body interactions by adjusting the frequencies of external potentials, the number of atoms, and the spatiotemporal modulated s-wave scattering lengths in real experiments.

In the same way as the study on BECs with spatiotemporally modulated two-body interactions [10, 17, 18], choosing different values of parameters $\omega$, $\sigma_1$, $\sigma_2$, $\nu_1$, $\nu_2$, $k$, and $\gamma$ in the exact solutions (22)-(23), several types of soliton behaviors, such as breathing solutions and quasi-periodic solutions are found and their dynamics is analyzed. Now we first provide the experimental parameters for producing the one-dimensional atom condensates composed of $N = 6 \times 10^4$ atoms with spatiotemporally modulated two- and three-body interactions, confined in a cigar-shaped trap with the ratio of the confining frequencies, $\omega = \omega_x/\omega_\perp$ of order $O(10^{-2})$. To be specific, we choose the axial frequency $\omega_x = 2\pi \times 70$ Hz and the radial frequency $\omega_\perp = 2\pi \times 710$ Hz. In what follows, we consider a more general external potential

$$V(x,t) = \frac{\omega^2}{2} x^2 - \frac{2\sigma_1 \delta^2}{\pi} e^{-2\delta^2 x^2},$$  \hspace{1cm} (24)
where

$$\omega^2 = \omega_0^2 + \epsilon \cos(2t), \quad \epsilon \ll 1. \quad (25)$$

It is remarked that when the parameter $\epsilon = 0$, the frequency of the external potential is $\omega = \omega_0 = \omega_x/\omega_\perp$. Figure 1 demonstrates the shape of this external potential for different parameters, which shows that it is a harmonic potential for parameter $\sigma_1 = 0$ and an anharmonic one, which is periodic in time $t$ for $\sigma_1 \neq 0$, where there are periodic humps $\sigma_1 < 0$ or periodic hollows $\sigma_1 > 0$ as time develops.

Moreover, it is known that the behavior of the scattering lengths near a Feshbach resonant magnetic field $B_0$ is typically of the form $a_s(B) = a[1 + \Delta/(B_0 - B)]$, with $a$ being the asymptotic value of the scattering length far from the resonance, $B_0$ being the resonant value of the magnetic field, and $\Delta$ being the width of the resonance. Here we consider the spatiotemporally modulated two- and three-body interactions, i.e., the scattering length $a_s$ is permitted to be time- and space-dependent ($a_s = a_s(x,t)$). Thus the magnetic field $B$ should vary with time and space [36]. Figures 2 and 3 show the spatiotemporal structures of the two- and three-body interactions $g_2$ and $g_3$ with parameters $\sigma_2 = 1, \omega = 7/80, \nu = K(k)$, and $k = 1/2$, where

$$K(s) = \int_0^{\pi/2} [1 - s^2 \sin^2 \tau]^{-1/2}d\tau.$$

Figure 2 demonstrates the attractive two- and three-body interactions $g_2$ and $g_3$ with $\sigma_1 = -2$ and $\sigma_3 = 0.7493$, and Fig. 3 demonstrates the attractive two-body and repulsive three-body interactions with $\sigma_1 = 2$ and
Localized nonlinear matter waves in one-dimensional Bose-Einstein condensates

3.1. BREATHING SOLUTIONS

We first consider the case of $\epsilon = 0$ in Eq. (25), i.e. the frequency of external potential is $\omega = \omega_0 = \omega_x/\omega_L$. Thus the ODE (16) has two linearly independent solutions $y_1 = \cos(\omega_0 t)$ and $y_2 = \sin(\omega_0 t)$ and the ODE (12) has a general exact solution

$$\delta = \left[a \cos(\omega_0 t)^2 + b \sin(\omega_0 t)^2 + 2c \cos(\omega_0 t) \sin(\omega_0 t)\right]^{-1/2}.$$ 

From the expressions of the exact localized nonlinear matter wave solutions (22) and (23), it is seen that the function $\sqrt{\delta}e^{\gamma dt}$ characterizes the amplitude and function $1/\delta$ characterizes the width of localized nonlinear matter waves. When considering the boundary condition $\lim_{|x| \to 0} \psi(x,t) = 0$ to get localized solutions, we derive immediately that $\nu_1 = (2n + 1)K(k)$ and $\nu_2 = 2mK(k)$ in Eqs. (22) and (23), where $K(s) = \int_0^{\pi/2} [1 - s^2 \sin^2 \tau]^{-1/2} d\tau$ is the elliptic integral of the first kind, and $n \geq 0$ and $m \geq 1$ are integers.

Figure 4 demonstrates the evolution of density distributions of the wave functions in Eqs. (22) and (23) with the distributed gain/loss effect $\gamma = 0$ and the parameters

$$\sigma_1 = 2, \quad \sigma_2 = 1, \quad \omega_0 = 7/80, \quad k = 1/2, \quad \sigma_3 = -0.0413, \quad b = 4, \quad c = 1, \quad \epsilon = 0.$$ 

(26)
Fig. 5 – (Color online) Dynamics of the collapsed breathing solutions (22) and (23) of the 1D cubic-quintic GP equation (6) with $\gamma = -0.003$. The other parameters are given in Eq. (26).

Fig. 6 – (Color online) Dynamics of the raised breathing solutions (22) and (23) of the 1D cubic-quintic GP equation (6) with $\gamma = 0.003$. The other parameters are given in Eq. (26).
In this case, the two-body interaction is attractive and the three-body interaction is repulsive as shown in Fig. 3. Figures 4(a) and 4(c) show the evolution of condensate density expressed by solution (22) with integer \( n = 0, 1 \), and Figs. 4(b) and 4(d) show the evolution of condensate density expressed by solution (23) with integer \( m = 1, 2 \). Figure 7(a) demonstrates the width \( 1/\delta \) and amplitude \( \sqrt{\delta} e^{\int \gamma dt} \) in this case. It is observed that we have obtained a series of localized nonlinear matter waves, which are space-localized and time-periodic breathing solutions of the 1D cubic-quintic GP equation (6). The widths and amplitudes of these localized nonlinear matter waves expand periodically with respect to time and keep their shapes invariant.

Fig. 7 – (Color online) The width \( 1/\delta \) (upper line) and the amplitude \( \sqrt{\delta} e^{\int \gamma dt} \) (lower line) of the wave functions in Eqs. (22) and (23). Here (a) demonstrates the case of \( \gamma = 0 \), (b) demonstrates the case of \( \gamma = -0.003 \), and (c) demonstrates the case of \( \gamma = 0.003 \).

Figures 5 and 6 demonstrate the evolution of density distributions of the wave functions in Eqs. (22) and (23) with the distributed gain/loss effect \( \gamma = -0.003 \) and \( \gamma = 0.003 \), respectively. The other parameters are the same as those in Fig. 4. Figures 7(b)-7(c) demonstrate the widths \( 1/\delta \) and amplitudes \( \sqrt{\delta} e^{\int \gamma dt} \) of the cases of \( \gamma = -0.003 \) and \( \gamma = 0.003 \), respectively. It is observed from Figs. 5-7 that the nonlinear matter waves we obtained are localized in space and collapse (\( \gamma = -0.003 \)) or raise (\( \gamma = -0.003 \)) in time. In the two cases, the widths of the localized nonlinear matter waves keep expanding periodically in time but the amplitudes collapse (\( \gamma = -0.003 \)) or raise (\( \gamma = -0.003 \)) as time develops.
838 Deng-Shan Wang, Yushan Xue, Zhifei Zhang 12

Fig. 8 – (Color online) Dynamics of the quasi-breathing solutions (22) and (23) for the 1D cubic-quintic GP equation (6) with \( \omega^2 = \omega_0^2 + \epsilon \cos(2t) \) and \( \gamma = 0 \). Here (a) and (c) show the evolution of condensate density given by solution (22) with \( n = 0, 1 \), and (b) and (d) show the evolution of condensate density given by solution (23) with \( m = 1, 2 \). The other parameters are \( \sigma_1 = 2, \sigma_2 = 1, \omega_0 = 7/80, k = 1/2, \sigma_3 = -0.0413, b = 4, c = 1, \) and \( \epsilon = 0.001 \).

3.2. QUASI-BREATHING SOLUTIONS

We now consider the case of \( \epsilon \neq 0 \) in Eq. (25), in which the ODE (16) has two linearly independent solutions described by the Mathieu functions \( y_1 = \text{MathieuC}(\omega_0^2, -\frac{\epsilon}{2}, t) \) and \( y_2 = \text{MathieuS}(\omega_0^2, -\frac{\epsilon}{2}, t) \) and the ODE (12) has a general exact solution

\[
\delta = [a \text{MathieuC}(\omega_0^2, -\frac{\epsilon}{2}, t)^2 + b \text{MathieuS}(\omega_0^2, -\frac{\epsilon}{2}, t)^2]^{1/2} + 2c \text{MathieuC}(\omega_0^2, -\frac{\epsilon}{2}, t) \text{MathieuS}(\omega_0^2, -\frac{\epsilon}{2}, t)]^{1/2}.
\]

In the same way, if we study the boundary condition \( \lim_{|x| \to 0} \psi(x, t) = 0 \) to get localized solutions, it is obtained immediately that \( \nu_1 = (2n + 1) K(k) \) and \( \nu_2 = 2mK(k) \) in Eqs. (22) and (23). In what follows, we only investigate the case of
the distributed gain/loss effect $\gamma = 0$. Figure 8 demonstrates the evolution of density distributions of the wave functions in Eqs. (22) and (23) with $\gamma = 0, \epsilon = 0.001$ and the parameters in Eq. (26). Figures 8(a) and 8(c) show the evolution of condensate density described by solution (22) with integer $n = 0, 1$, and Figs. 8(b) and 8(d) show the evolution of condensate density described by solution (23) with integer $m = 1, 2$. It can be observed that the localized nonlinear matter waves we obtain are time quasi-periodic breathing solutions compared with the breathing solutions obtained above. Moreover, the widths and amplitudes of nonlinear matter waves also develop quasi-periodically in time.

Finally, it is remarked that when choosing the parameters $\sigma_1 = -2, \sigma_2 = 1, \sigma_3 = 0.7493$, i.e. both the two-body interaction and the three-body interaction are attractive, we can also get the same density distributions of the wave functions in Eqs. (22) and (23) as shown in Figs. 4-8. This denotes that both attractive two- and three-body interactions and attractive two-body plus repulsive three-body interactions support localized nonlinear matter waves in BEC with spatiotemporally modulated interactions and trapped in time-dependent external potentials. Moreover, it is seen from Figs. 4-8 that the density distribution based on wave function (22) has $2n + 1$ wave profiles in the space direction for fixed time $t$, and the density distribution based on wave function (23) has $2m$ wave profiles in the space direction for fixed time $t$, where $n \geq 0$ and $m \geq 1$ are integers.

4. CONCLUSION

In conclusion, two exact localized nonlinear matter wave solutions of the 1D cubic-quintic GP equation (6) with spatiotemporally modulated interactions and time-dependent external potential are proposed explicitly. The dynamics of these localized nonlinear matter waves is analyzed by choosing different physical parameters. It is shown that spatiotemporal modulated two- and three-body interactions can support novel localized nonlinear matter waves in cubic-quintic GP equation. This work can stimulate further studies of nonlinear matter waves in systems with spatiotemporal modulated nonlinearities and time-dependent external potentials.

Acknowledgements. This work was supported by National Natural Science Foundation of China under Grant Nos. 11271362 and 11375030, Beijing Natural Science Fund Project and Beijing City Board of Education Science and Technology Key Project No. KZ201511232034, Beijing Natural Science Foundation under Grant No. 1153004, Beijing Nova program No. Z131109000413029, Beijing Finance Funds of Natural Science Program for Excellent Talents No. 2014000026833ZK19, and 2016 Beijing merit funding for study abroad scholars in science and technology activities.
REFERENCES

34. V. P. Ermakov, Univ. Izv. Kiev. 20, 1-19 (1880).